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SOME OPTIMIZATION PROBLEMS RELATED TO COOLING FINS

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Abstract—The problems of minimizing the volume of purely conducting and conducting–convecting fins are solved. Exact solutions are obtained for the corresponding cross sectional areas and the temperature distributions. An approximate solution is also given for a convecting–radiating fin. The results are plotted and discussed.

NOMENCLATURE

a, cross sectional area of fin;
a_0, cross sectional area of fin at $x = 0$;
B_i, Biot number;
h, heat-transfer coefficient;
J, heat flux at the base of the fin;
k, thermal conductivity;
L, length of fin;
n, h/k;
N_G, Generation number;
q_0, uniform heat source distribution;
q, heat source distribution;
T, temperature;
T_m, ambient temperature;
T_0, temperature at the base of fin, $x = 0$;
V, volume of fin;
x, length of fin;
σ, Stefen–Boltzmann constant;
λ, Langrangian multiplier;
θ, temperature, $T - T_m$;
ε, emissivity;
γ, $\varepsilon\sigma/k$.

1. INTRODUCTION

THE FIRST extensive study of the heat transfer from fins was carried out by Harper and Brown [1]. Mathematical fin theory have been discussed by Jakobi [2] and Schneider [3] among others. For pure conducting fins, a criterion for optimum shape was proposed by Schmidt [4] which was later proved by Duffin [5]. Optimization of a rectangular fin was studied by Liu [6]. The effect of internal heat generation on the optimal shape was first considered by Minker and Rouleau [7] but a more rigorous treatment was given by Liu [8] for heat generations which are directly proportional to the temperature. Optimum shape of a purely radiating fin was obtained by Wilkins [9–11]. Recently, approximate optimum fin design for boiling heat transfer was considered by Cash *et al.* [12]. In the present work some optimal design problems in heat transfer are treated. In the sections that follow first a purely conducting fin with arbitrary heat source distribution is considered. Exact solutions are obtained for the cross sectional area and the corresponding temperature field which minimizes the maximum temperature in the domain.

The problem is then extended to a conducting–convecting fin with an arbitrary heat source. For a given rate of heat transfer a variational problem is set up which minimizes the volume of the fin. Exact solutions are found for the special case of uniform source distribution. Finally the case of a convecting–radiating fin is studied. For small rate of radiative transfer the equation of heat transfer is linearized and approximate optimal thickness is obtained. The results are plotted and discussed.

2. MINIMIZING THE MAXIMUM TEMPERATURE

Consider an arbitrary heat generation in a thin region with one end insulated and the other end is kept at constant temperature. The optimization problem is to find the cross sectional area in such a way that the maximum temperature which occurs at the insulated boundary becomes a minimum for a fixed volume of the conducting material.

Assuming that the region is sufficiently thin so that the one dimensional approximation is valid, the equation of heat conduction becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[a(x)\frac{\mathrm{d}T}{\mathrm{d}x}\right] = -q(x) \tag{1}$$

where $a(x)$ is the cross sectional area and $q(x)$ is the heat source distribution divided by the thermal conductivity of the medium. The total volume of the material is assumed to be fixed, i.e.

$$V = \int_0^L a(x)\,\mathrm{d}x, \tag{2}$$

where L is the length of fin. The boundary conditions are

$$T(0) = 0, \tag{3a}$$

$$\frac{\mathrm{d}T}{\mathrm{d}x}(L) = 0. \tag{3b}$$

In order to minimize the maximum temperature it is advantageous to find the formal solution of differential equation (1) under the boundary conditions (3). Integrating equation (1) from L to x and using the boundary condition (3b) we find

$$a(x)\frac{\mathrm{d}T}{\mathrm{d}x} = \int_x^L q(\alpha)\,\mathrm{d}\alpha. \tag{4}$$

Dividing equation (4) by $a(x)$ and integrating once more from 0 to x and making use of the boundary condition (3a) yields,

$$T(x) = \int_0^x \frac{1}{a(\beta)}\int_\beta^L q(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{5}$$

From the above it is obvious that the maximum temperature occurs at the insulated boundary, i.e.

$$T(L) = \int_0^L \frac{Q(x)}{a(x)}\mathrm{d}x, \tag{6}$$

where

$$Q(x) = \int_x^L q(\alpha)\,\mathrm{d}\alpha. \tag{7}$$

$Q(x)$ represents the total heat generation from the insulated boundary up to the point x.

The extremization problem then is to find the minimum of (6) under the integral equality constraint (2). This is a simple problem in calculus of variation [13]. Accordingly,

$$\delta\int_0^L\left[\frac{Q(x)}{a(x)} + \lambda a(x)\right]\mathrm{d}x = 0 \tag{8}$$

where λ is a constant Lagrangian multiplier and δ denotes the variation operator. Euler's equation for the variational problem (8) is simply,

$$-\frac{Q(x)}{a^2} + \lambda = 0, \tag{9}$$

or

$$a(x) = \left(\frac{Q(x)}{\lambda}\right)^{\frac{1}{2}}. \tag{10}$$

The constant λ is easily obtained by direct substitution of (10) in equation (2). The optimal cross sectional area then becomes

$$a(x) = V[Q(x)]^{\frac{1}{2}}/\int_0^L [Q(x)]^{\frac{1}{2}}\,\mathrm{d}x. \tag{11}$$

The temperature distribution may then be obtained from (5).

As an example let us consider the case of a heat source distribution of the form

$$q = q_0(1 - x/L)^M, \qquad M \geqslant 0 \tag{12}$$

For this case we have

$$Q(x) = \frac{q_0 L}{M+1}(1 - x/L)^{M+1}$$

$$a(x) = \frac{(M+3)V}{2L}(1 - x/L)^{(M+1)/2} \tag{13}$$

$$T(x) = \frac{4q_0 L^3}{(M+1)(M+3)^2 V} \times [1 - (1 - x/L)^{(M+3)/2}].$$

The maximum temperature then becomes

$$T(L) = \frac{4q_0 L^3}{(M+1)(M+3)^2 V} \tag{14}$$

comparison of equation (14) with the maximum temperature in a uniform cross section fin gives a reduction factor of $[(M+1)/(M+3)]^2$.

For the special case of $M = 0$, that is a uniform heat source distribution the above reduces to

$$q = q_0$$

$$a(x) = \frac{3V}{2L}(1 - x/L)^{\frac{1}{2}} \tag{15}$$

$$T(x) = \frac{4q_0 L^3}{qV}[1 - (1 - x/L)^{\frac{3}{2}}]$$

and

$$T(L) = \frac{4q_0 L^3}{9V}. \tag{16}$$

Comparing the above with the maximum temperature in a uniform cross section fin we note that the reduction is about 11 per cent.

The optimal fin cross sections for various types of heat sources are plotted in Fig. 1. It is observed that the optimal cross sectional areas for $M < 1$ are concave and for $M > 1$ are convex. For the special case of constant heat generation, the optimal temperature profile and the temperature distribution of a constant area fin are plotted in Fig. 2. It is interesting to point out that the optimal temperature is less than the uniform case everywhere in the fin.

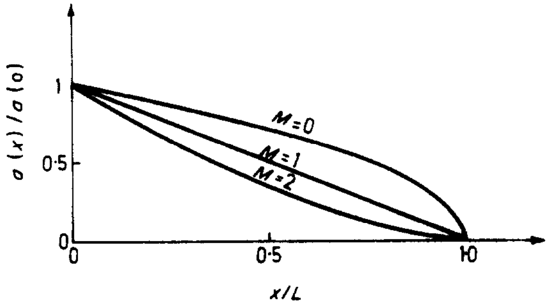

FIG. 1. The optimal cross sectional areas for various heat generations.

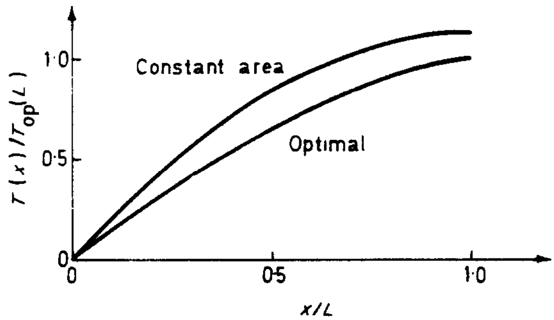

FIG. 2. The temperature distributions in the optimal fin and a constant area fin.

3. MINIMUM WEIGHT DESIGN OF A CONVECTING FIN

In the present section we are concerned with the minimum weight design of a fin with arbitrary heat source distribution and constant heat flux at the base. With no loss in generality the width of the fin is chosen to be 1 ft.

The equation governing the heat transfer in a thin fin is

$$\frac{d}{dx}\left[a(x)\frac{dT}{dx}\right] - n(T - T_m) + q(x) = 0, \tag{17}$$

$$n = h/k, \tag{18}$$

where h is heat transfer coefficient, k is the thermal conductivity, and T_m is the temperature of the ambient fluid. The boundary conditions are

$$a(x)\left.\frac{dT}{dx}\right|_{x=0} = -J, \tag{19a}$$

$$T|_{x=0} = T_0, \tag{19b}$$

$$T|_{x=L} = T_m. \tag{19c}$$

The boundary condition (19c) implies that the tip of the fin is at the same temperature as the surrounding medium. This has been discussed by Minkler and Rouleau [7] and was obtained by Liu [8] through his variational scheme. In order to simplify the problem let

$$\theta = T - T_m. \tag{20}$$

Equation (17) and the boundary conditions (19) then become

$$\frac{d}{dx}\left[a(x)\frac{d\theta}{dx}\right] - n\theta + q(x) = 0, \tag{21}$$

and

$$a(x)\left.\frac{d\theta}{dx}\right|_{x=0} = -J, \tag{22a}$$

$$\theta|_{x=0} = T_0 - T_m = \theta_0, \tag{22b}$$

$$\theta|_{x=L} = 0. \tag{22c}$$

The optimization problem is the following: For a given heat flux $-J$ per unit width at the base of the fin, the thickness $a(x)$ is to be determined in such a way that the total volume

$$V = \int_0^L a(x)\,dx \tag{23}$$

becomes a minimum. In order to put the above optimization problem in variational form we rewrite the second order differential equation (21) in terms of two first order differential equations constraints, i.e.

$$\theta' = \eta \tag{24}$$

$$(a\eta)' - n\theta + q = 0 \tag{25}$$

where prime denotes differentiation with respect to x. Introducing the Langrangian multipliers $\lambda_1(x)$ and $\lambda_2(x)$ the extremization condition becomes,

$$\delta\int_0^L [a + \lambda_1(\theta' - \eta) + \lambda_2(a\eta' + a'\eta - \theta + q)]\,dx = 0. \tag{26}$$

The Euler–Lagrange equations satisfying (26) are

$$1 + \lambda\eta_2' - \frac{d}{dx}(\lambda_2\eta) = 0 \tag{27}$$

$$-\lambda_1 + \lambda_2 a' - \frac{d}{dx}(\lambda_2 a) = 0 \tag{28}$$

$$-n\lambda_2 - \frac{d\lambda_1}{dx} = 0. \tag{29}$$

The solutions of the set of differential equations (27)–(29) together with (21) and boundary conditions (22) give the optimal thickness and the corresponding temperature field. Eliminating λ_1 and λ_2 between equations (27)–(29) we find

$$n\theta'^2 + a\theta'\theta''' - a''\theta'^2 - 2\theta''^2 a + 2a''\theta'\theta'' = 0. \tag{30}$$

Equation (21) with the aid of boundary condition (23a) may be integrated once to give

$$a(x) = \frac{1}{\theta'}\left\{\int_0^x [n\theta(\alpha) - q(\alpha)]\,d\alpha - J\right\}. \tag{31}$$

Differentiating equation (21) and subtracting from (30) yields

$$2\theta''^2 a = 2n\theta'^2 - 2a''\theta'^2 - \theta' q'. \tag{32}$$

It is now possible to apply a numerical technique by first assuming a $\theta'(0)$ and integrating step by step up to the point $x = L$ and check to see if θ becomes zero and if not correcting $\theta'(0)$ and repeating the integration. This procedure is well known for solving two points boundary value problems [14].

If the heat source distribution is constant, that is

$$q(x) = q_0, \tag{33}$$

it is possible to find exact solutions,

$$a = +\frac{LJ}{\theta_0} - n\left(Lx - \frac{x^2}{2}\right) + \frac{q_0 Lx}{\theta_0}. \tag{34}$$

$$\theta = \frac{\theta_0}{L}(L - x). \tag{35}$$

Expressions (34) and (35) are the optimal thickness and the corresponding temperature field in the fin respectively. These are valid as long as expression (34) remain positive.

The areas of the base and the tip of the optimal fin are

$$a_0 = \frac{LJ}{\theta_0} \tag{36}$$

$$a_L = \frac{LJ}{\theta_0} - n\frac{L^2}{2} + \frac{q_0 L^2}{\theta_0}. \tag{37}$$

The total volume per unit width of the material used in the optimal fin is

$$V = \frac{L^2 J}{\theta_0} - n\frac{L^3}{3} + \frac{q_0 L^3}{2\theta_0}. \tag{38}$$

Introducing dimensionless numbers

$$B_i = \frac{nL^2}{a_0} = \frac{hL^2}{ka_0} = \text{Biot number} \tag{39}$$

$$N_G = \frac{q_0 L^2}{a_0 \theta_0} = \text{Generation number}. \tag{40}$$

The optimal cross sectional area becomes

$$a/a_0 = 1 - B_i[x/L - 0{\cdot}5(x/L)^2] + N_G x/L. \tag{41}$$

Figure 3 shows plots of the optimal areas for $B_i = 2$ and various Generation numbers. The temperature distribution in a uniform thickness fin is

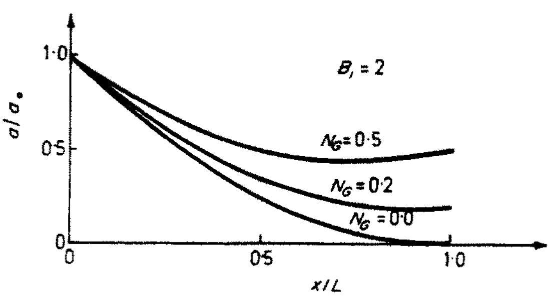

FIG. 3. The optimal cross sectional areas for various generation numbers.

$$\theta = \left[\theta_0 - \frac{q_0}{n}(1 - \cosh\sqrt{(n/a)})\right] \frac{\sin h\sqrt{(n/a_0)}(L - x)}{\sin h\sqrt{(n/a_0)}L} + \frac{q_0}{n}[1 - \cosh\sqrt{(n/a)}(L - x)]. \tag{42}$$

Defining the effectivness of the optimal fin as

$$\eta = \frac{[\text{Heat flux/volume] of optimal fin}}{[\text{Heat flux/volume] of uniform fin}}$$

we find

$$\eta = \frac{\tanh\sqrt{(B_i)}}{\sqrt{(B_i)}(1 - B_i/3 + N_G/2)[1 - (N_G/B_i)(1 - 1/\cosh(\sqrt{B_i}))]}. \tag{43}$$

Figure 4 shows the effectivness of the optimal fin for various Biot numbers and generation

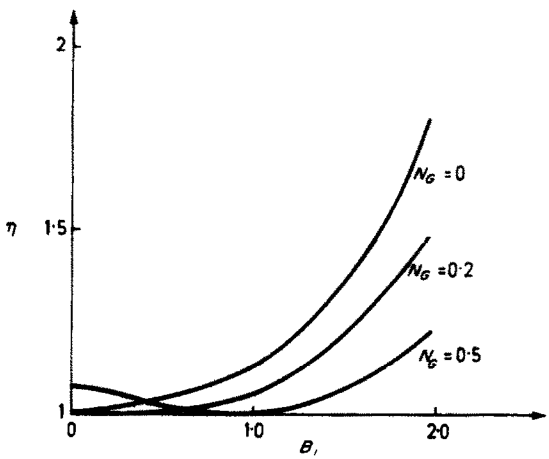

FIG. 4. The effectiveness of the optimal fin for various Biot numbers and Generation numbers.

numbers. It is observed that for a wide range of B_i and N_G the reduction in the volume of the optimal fin is quite significant.

4. APPROXIMATE MINIMUM WEIGHT DESIGN OF A CONVECTING–RADIATING FIN

The equation of heat transfer for a convecting–radiating fin is

$$\frac{d}{dx}\left(a(x)\frac{dT}{dx}\right) - n(T - T_m) - \gamma(T^4 - T_m^4) + q = 0 \quad (44)$$

where

$$\gamma = \frac{\varepsilon\sigma}{k} \quad (45)$$

with σ and ε being the Stefan–Boltzmann constant and emissivity, respectively. The boundary conditions are the same as that of the previous case. The optimization problem again is to find $a(x)$ in such a way that the volume given by equation (23) become a minimum.

An exact solution of the above optimal design problem is given by Wilkins [11] for the special case of a purely radiating fin to absolute zero. Although this solution is exact it has limited application due to its highly restrictive assumptions.

In the present work we intend to find an approximate solution for the special case when heat transfer by radiation is much smaller than convection. The approximation lies in the linearization of the nonlinear radiation term in (44). We write

$$\begin{aligned} T^4 - T_m^4 &= (T - T_m)(T^3 + T^2T_m + TT_m^2 + T_m^3) \\ &= (T - T_m)f(T_0, T_m) \end{aligned} \quad (46)$$

where

$$f(T_0, T_m) = \frac{(T_0 + T_m)^3}{8} + \frac{(T_0 + T_m)^2}{4}T_m + \frac{(T_0 + T_m)}{2}T_m^2 + T_m^3. \quad (47)$$

Introducing the change of variable (20) the linearized equation of heat transfer becomes

$$\frac{d}{dx}\left[a(x)\frac{d\theta}{dx}\right] - n_r\theta + q(x) = 0 \quad (48)$$

where

$$n_r = n + \gamma f(T_0, T_m). \quad (49)$$

The equation (48) is similar to (21) with n being replaced by n_r. Therefore the optimal thickness and temperature are given by

$$a(x) = \frac{LJ}{\theta_0} - n_r\left(Lx - \frac{x^2}{2}\right) + \frac{q_0Lx}{\theta_0} \quad (50)$$

$$\theta = \frac{\theta_0}{L}(L - x). \quad (51)$$

Comparing (50) with (34) it is observed that the optimal fin is thinner when the radiation is included.

The analysis in this section is restricted to the small rate of radiative transfer.

REFERENCES

1. D. R. Harper and W. B. Brown, Mathematical equations for heat conduction in the fins of air cooled engines, Nation Advisory Commission, Aeronautics, Report No. 158 (1922).
2. M. Jakobi, *Heat Transfer*, vol. I. Wiley, New York (1949).
3. P. J. Schneider, *Conduction Heat Transfer*. Addison-Wesley, Reading, Mass. (1955).
4. E. Schmidt, Die Wärmeübertragung durch Rippen. *Z. Deutsch Ing.* **70**, 885–947 (1926).
5. R. J. Duffin, A variational problem relating to cooling fins, *J. Math. Mech.* **8** (1), 47 (1959).
6. C. Y. Liu, On optimum rectangular cooling fins, *Q. Appl. Math.* **XIX** (1), 72–75 (1961).
7. W. S. Minkler and W. T. Rouleau, The effect of internal heat generation on heat transfer in thin fins, *Nucl. Sci. Engng* **7**, 400–406 (1960).
8. C. Y. Liu, A variational problem relating to cooling fins with heat generation, *Q. Appl. Math.* **XIX** (3), 245–251 (1961).
9. J. E. Wilkins, Jr., Minimizing the mass of thin radiating fins, *J. Aerospace Sci.* **27** (2), 145 (1960).
10. J. E. Wilkins, Jr., Minimum mass thin fins for space radiators, proceedings 1960, Heat transfer and Fluid Mechanics Institute. Stanford University Press (1960).
11. J. E. Wilkins, Jr., Minimum mass thin fins which transfer heat only by radiation to surrounding at absolute zero, *J. Soc. Ind. Appl. Math.* **8**, 630–639 (1960).
12. D. R. Cash, C. J. Klein and J. W. Westwater, Approximate optimum fin design for boiling heat transfer, *J. Heat Transfer* **93** (1), 19–24 (1970).
13. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part I. McGraw-Hill, New York (1953).
14. L. Fox, *The Numerical Solution of Two-Point Boundary Value Problem in Ordinary Differential Equation*. Oxford University Press (1957).

QUELQUES PROBLEMES D'OPTIMISATION RELATIFS AUX AILETTES DE REFROIDISSEMENT

Résumé—On traite des problèmes de minimisation du volume des ailettes conductrices et convectantes. On obtient des solutions exactes correspondant à des formes de sections droites et à des distributions de température. Une solution approchée est donnée pour une ailette en rayonnement et convection. Les résultats sont représentés graphiquement et discutés.

OPTIMIERUNGSPROBLEME VON KÜHLRIPPEN

Zusammenfassung—Die Probleme der Optimierung des Volumens von Rippen mit reiner Leitung und mit Leitung und Konvektion werden gelöst. Exakte Lösungen erhält man für die einander entsprechenden Querschnittsflächen und Temperaturverteilungen. Eine Näherungslösung ist angegeben für eine Rippe mit Konvektion und Strahlung. Die Ergebnisse werden grafisch dargestellt und diskutiert.

НЕКОТОРЫЕ ЗАДАЧИ ОПТИМИЗАЦИИЙ, СВЯЗАННЫЕ С ОХЛАЖДАЮЩИМИ РЕБРАМИ

Аннотация—Решаются задачи минимизации объема ребер в случаях чистой теплопроводности или при наличии конвекции. Получены точные решения для соответствующих площадей поперечного сечения и распределений температуры. Приводится также приближенное решение для ребра в случае совместной конвекции и излечения. Проведено обсуждение результатов, которые построены графически.